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SPECTRAL PROPERTIES AND LENGTH SCALES OF TWO-DIMENSIONAL MAGNETIC FIELD MODELS

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ABSTRACT

Two-dimensional (2D) models of magnetic field fluctuations and turbulence are widely used in space, astrophysical, and laboratory contexts. Here we discuss some general properties of such models and their observable power spectra. While the field line random walk in a one-dimensional (slab) model is determined by the correlation scale, for 2D models, it is characterized by a different length scale, the ultrascale. We discuss properties of correlation scales and ultrascales for 2D models and present a technique for determining an ultrascale from observations at a single spacecraft, demonstrating its accuracy for synthetic data. We also categorize how the form of the low-wavenumber spectrum affects the correlation scales and ultrascales, thus controlling the diffusion of magnetic field lines and charged test particle motion.

Subject headings: diffusion — magnetic fields — turbulence

1. INTRODUCTION AND BACKGROUND

Two-dimensional (2D) models have been extensively studied in magnetohydrodynamic (MHD) turbulence (Fyfe & Montgomery 1976; Fyfe et al. 1977) and appear as an asymptotic limit of anisotropic low-frequency MHD turbulence in a regime known as reduced MHD (RMHD; Strauss 1976; Montgomery 1982; Higdon 1984; Goldreich & Sridhar 1995). Quasi–two-dimensional, RMHD, or related models appear in descriptions of laboratory plasma behavior and coronal magnetic fields and lie at the core of reconstruction techniques (Hu & Sonnerup 2003) based on the Grad-Shafronov equation. Quasi-2D and RHMD models are characterized by gradients in the direction along the mean magnetic field that are weak compared to gradients in the transverse directions. In interplanetary studies, there is evidence that the magnetic field fluctuations admit a strong component of nearly two-dimensional character, comprising perhaps 70%–90% of the turbulent inertial range energy budget (Bieber et al. 1996). This has leading order consequences for the scattering properties of cosmic rays of solar (Bieber et al. 1994) and Galactic (Burger & Hattingh 1998) origin. The magnetic field in the plasma sheet of the Earth's magnetotail is also believed to have a quasi-2D structure (Montgomery 1987; Borovsky et al. 1997).

Motivated by these applications, each of which may reference, in certain approximations, the detailed properties of two-dimensional turbulent magnetic (or velocity) fields, we examine here some distinctive properties of the power spectra of 2D turbulent, divergence-free fields. We then examine the nature of the correlation scales and ultrascales, the latter being distinct scales that determine the field line diffusion and particle transport properties. As both of these types of length scales are sensitive to the long-wavelength behavior of the power spectrum, the nature of the structures at energy-containing and larger scales have important quantitative effects on transport phenomena in astrophysical plasmas with quasi-2D turbulence. To this end, we categorize the low-wavenumber spectrum, including constraints from homogeneity requirements, and determine how it affects the correlation scale and ultrascale. Furthermore, we provide a technique for determining the ultrascale from observations, which has been successfully tested for simulated data

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2. BASIC PROPERTIES OF 2D MODELS

We consider models in which the total magnetic field $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{B}_0 + \boldsymbol{b}(\boldsymbol{x})$ is the sum of a uniform mean field $\boldsymbol{B}_0$ and a fluctuation $\boldsymbol{b}$ that is perpendicular to the mean, $\boldsymbol{b} \cdot \boldsymbol{B}_0 = 0$. Furthermore, let the fluctuation field depend only on the two transverse coordinates, so that

$$\boldsymbol{b}(x, y) = \nabla \times a(x, y)\hat{\boldsymbol{z}} = (\partial a/\partial y, -\partial a/\partial x, 0), \tag{1}$$

where $a\hat{\boldsymbol{z}}$ can be interpreted as a vector potential for the 2D component of turbulence, and $a(x, y)$ is a potential function or a poloidal (transverse) flux function, in the sense that $\int_1^2 \boldsymbol{b}^{2\mathrm{D}} \cdot \hat{\boldsymbol{n}}\, d\ell = a(2) - a(1)$, where $d\ell$ is the line element along any curve connecting points 1 and 2, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{t}} \times \hat{\boldsymbol{z}}$ is the 2D normal to that curve, and $\hat{\boldsymbol{t}}$ is the unit vector tangent to $d\ell$. We do not address the dynamical situations that give rise to two dimensionality, but rather examine some of its properties that might be relevant in applications.

The term "two and a half dimensional" fluctuations usually refers to the case in which a parallel component of the fluctuations is also present, but depending only on the transverse coordinates, so that $\boldsymbol{b}^{2.5\mathrm{D}}(x, y) = (\partial a/\partial y, -\partial a/\partial x, b_z(x, y))$. However, for most of what follows, we discuss the 2D model (eq. [1]).

2.1. *Periodic and Unbounded Representations*

As in treatments of fully three-dimensional turbulence, it is useful to work sometimes in a large but finite periodic box of side $2\pi L$, in which functions, such as the potential function, can be expressed in a Fourier series. For example, we have $a(x, y) = \Sigma_{\boldsymbol{k}} \tilde{a}_L(k_x, k_y) \exp(i\boldsymbol{k} \cdot \boldsymbol{x})$, with the sum running over $k_x, k_y = 0, \pm k_0, \pm 2k_0 \ldots$, where $k_0 = 1/L$ is the smallest allowed nonzero wavenumber, and therefore the wavenumber spacing. To pass to the homogeneous (infinite) system limit, all physical scales are held fixed while the box size increases ($L \rightarrow \infty$). In this way, for example, the spectrum of the potential function $\tilde{A}_L(\boldsymbol{k}) = \langle |\tilde{a}_L(\boldsymbol{k})|^2 \rangle$ in the periodic domain of size L is asymptotically related in the usual way to the homogeneous spectral density $A(\boldsymbol{k}) = \lim_{L\rightarrow\infty} (L)^d \tilde{A}_L(\boldsymbol{k})$. Here the dimensionality is $d = 2$, and the brackets $\langle \ldots \rangle$ denote an ensemble average or equivalent time-space average. With this limit understood, this allows us to freely pass between the periodic and homogeneous cases,

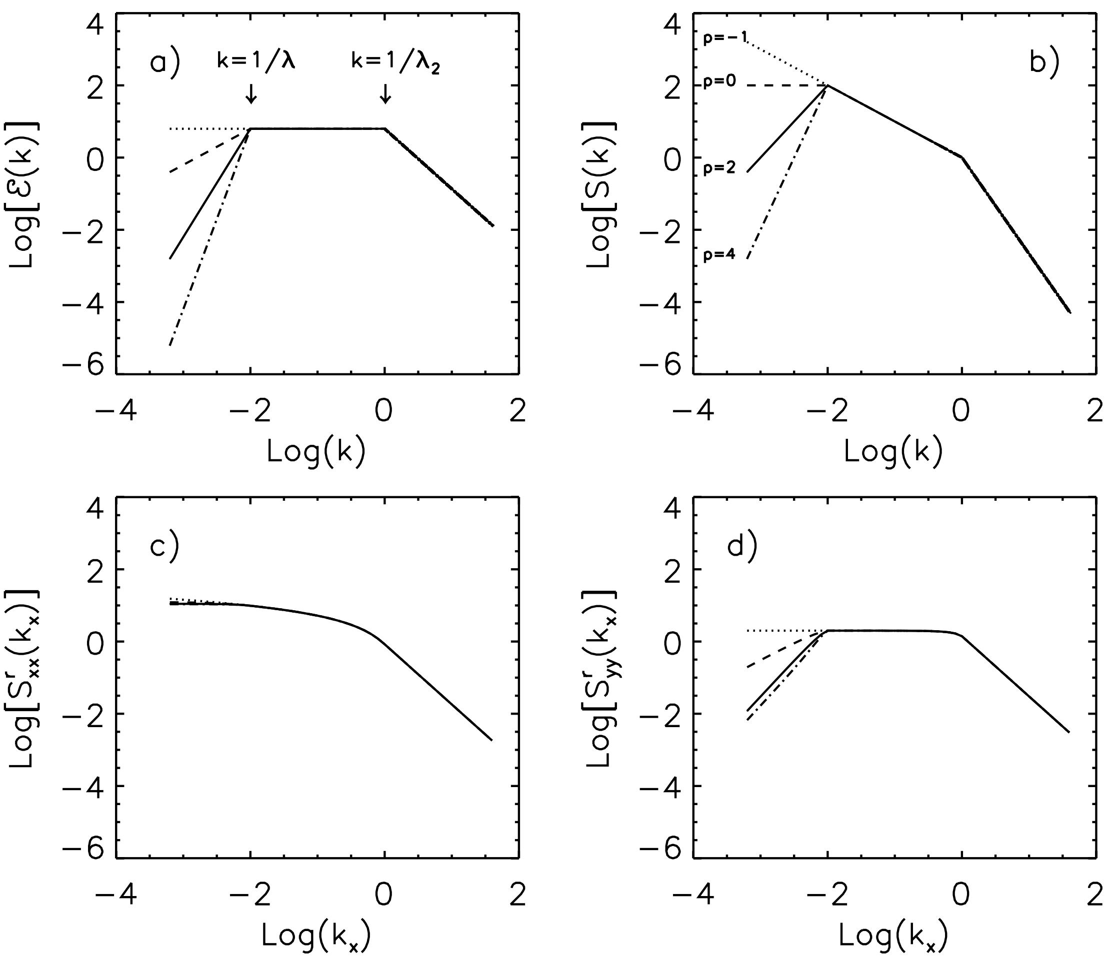

FIG. 1.—Example of axisymmetric 2D magnetic power spectra with two length scales: λ_2, the 2D bendover scale, and λ, which marks the onset of long-wavelength (low-wavenumber) behavior. Here $S(k)$ is specified by eq. (26) for $p = -1$ (*dotted lines*), $p = 0$ (*dashed lines*), $p = 2$ (*solid lines*), and $p = 4$ (*dash-dotted lines*), with $C\langle b^2\rangle = 1$, $\nu = -5/3$, $\lambda_2 = 1$, and $\lambda = 100$. (*a*) Omnidirectional energy spectrum, $\mathcal{E}(k)$; (*b*) modal spectrum, $S(k)$; (*c*) longitudinal reduced spectrum, $S^r_{xx}(k_x)$; and (*d*) transverse reduced spectrum, $S^r_{yy}(k_x)$.

and a symbol with a tilde (˜) designates the periodic counterpart. Below, the wavevector is understood as two-dimensional, $\boldsymbol{k} = (k_x, k_y, 0)$, unless stated otherwise. Note that the spectrum of 2D fluctuations can be embedded conveniently in a full three-dimensional spectrum according to $S^{3\mathrm{D}}(\boldsymbol{k}) = S(k_x, k_y)\delta(k_z)$.

In the periodic domain and using $\tilde{\boldsymbol{b}}_L = i\boldsymbol{k} \times \hat{z}\tilde{a}_L(\boldsymbol{k})$, one easily sees that $\tilde{S}^L_{xx}(\boldsymbol{k}) = \langle|\tilde{b}_x L(\boldsymbol{k})|^2\rangle = k_y^2\tilde{A}_L(\boldsymbol{k})$ and $\tilde{S}^L_{yy}(\boldsymbol{k}) = \langle|\tilde{b}_y L(\boldsymbol{k})|^2\rangle = k_x^2\tilde{A}_L(\boldsymbol{k})$. This corresponds to the expected homogeneous symmetric spectral tensor (Batchelor 1970)

$$S_{ij}(\boldsymbol{k}) = \left(\delta^{(2)}_{ij} - \frac{k_i k_j}{k^2}\right)S(\boldsymbol{k}), \tag{2}$$

where $\delta^{(2)}_{ij}$ is the identity tensor in the (x, y) plane, and S denotes the modal spectral density $S(\boldsymbol{k}) = (k_x^2 + k_y^2)A(\boldsymbol{k}) = k^2 A(\boldsymbol{k})$. The solenoidal condition is explicitly satisfied: $k_i S_{ij} = k_i S_{ji} = 0$.

2.2. *Reduced Spectra*

Each of the spectra is the Fourier transform of a corresponding two-point correlation function. That is, defining $R_{ij}(\boldsymbol{r}) = \langle b_i(\boldsymbol{x})b_j(\boldsymbol{x} + \boldsymbol{r})\rangle$, the spectra are

$$S_{ij}(\boldsymbol{k}) = (2\pi)^{-3}\int d^3r\, R_{ij}(\boldsymbol{r})\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}). \tag{3}$$

Here $\boldsymbol{x}$ and $\boldsymbol{r}$ are, respectively, the (arbitrary) absolute position vector and the lag vector in the (x, y) plane. Usually measurements are made along a line, so that only quantities such as $R(x, 0)$ or $R(0, y)$ are known experimentally. This would be the case in a wind tunnel or using instrumental data from a single spacecraft in the solar wind. The Fourier transforms of these measured correlation functions, known as reduced spectra, are therefore frequently relevant. We also define $R = R_{xx} + R_{yy}$ and $\langle b^2\rangle = \langle b_x^2\rangle + \langle b_y^2\rangle$.

Consider the reduced spectrum associated with $R(x, 0)$, i.e., for measurement locations displaced only along the x-direction. Then there are two independent reduced spectra, the parallel or longitudinal spectrum,

$$S^r_{xx}(k_x) = \int_{-\infty}^{\infty} dk_y\left(1 - \frac{k_x^2}{k^2}\right)S(k_x, k_y) = \int_{-\infty}^{\infty} dk_y k_y^2 A(\boldsymbol{k}), \tag{4}$$

and the perpendicular or transverse spectrum,

$$S^r_{yy}(k_x) = k_x^2\int_{-\infty}^{\infty} dk_y A(\boldsymbol{k}). \tag{5}$$

Examples of reduced spectra are shown in Figure 1.

An important relationship exists between these two when 2D turbulence is axisymmetric. In that case, $A(\boldsymbol{k}) = A[(k_x^2 + k_y^2)^{1/2}]$, and thus $k_x^{-1}\partial A/\partial k_x = k_y^{-1}\partial A/\partial k_y = k^{-1}\partial A/\partial k$. Using this, one immediately sees that $dS_{xx}^r(k_x)/dk_x = k_x \int dk_y k_y\, dA/dk_y = -k_x \int A\, dk_y$, with the latter step following after integration by parts. Therefore, the two reduced spectra are related by

$$-k_x \frac{dS_{xx}^r(k_x)}{dk_x} = S_{yy}^r(k_x). \tag{6}$$

This has important observational consequences. In standard time series analysis, or for so-called slab turbulence (with $b_{x,y}$ varying only in z), one becomes accustomed to observed reduced spectra that "bend over" at some wavenumber $1/\lambda$, and for lower wavenumbers, the spectrum becomes flat at a level of order $S \sim \langle b^2 \rangle \lambda$ (see more on this below). However, for pure 2D turbulence, this cannot occur for both observed spectra. From equation. (6), if $S_{xx}^r \sim$ constant at low k_x, then $S_{yy}^r \to 0$ in that range of scales (as in Fig. 1, for $p = 0$, 2, and 4). Likewise, if $S_{yy}^r \sim$ constant at low k_x, then S_{xx}^r logarithmically diverges as $k_x \to 0$ (as in Fig. 1, for $p = -1$). This behavior is related to the nature of the large closed islands that can exist in 2D turbulence.

3. CORRELATION SCALES AND ULTRASCALES

In the classical definitions of homogeneous turbulence (Batchelor 1970), the correlation scale (or integral scale) is associated with the area under the correlation function. Just as there are two reduced spectra, there may be two correlation scales in 2D turbulence, again depending on whether the Cartesian component involved is in the direction of integration or perpendicular to it. Thus the longitudinal correlation scale is

$$\begin{aligned}\lambda_{c\parallel} &= \frac{\int_0^\infty R_{xx}(x,0)\,dx}{R_{xx}(0,0)} = \frac{\pi}{\langle b_x^2 \rangle} S_{xx}^r(k_x)|_{k_x=0} \\ &= \frac{\pi}{\langle b_x^2 \rangle} \int_{-\infty}^{\infty} k_y^2 A(k_x = 0, k_y)\,dk_y\end{aligned} \tag{7}$$

and the lateral, or transverse, correlation scale is

$$\begin{aligned}\lambda_{c\perp} &= \frac{\int_0^\infty R_{yy}(x,0)\,dx}{R_{yy}(0,0)} = \frac{\pi}{\langle b_y^2 \rangle} S_{yy}^r(k_x)|_{k_x=0} \\ &= \frac{\pi}{\langle b_y^2 \rangle} \left[k_x^2 \int_{-\infty}^{\infty} A(k_x, k_y)\,dk_y \right]_{k_x=0}.\end{aligned} \tag{8}$$

From the last expression it is clear that $\lambda_{c\perp} \equiv 0$, unless the integral $I = \int dk_y A(0, k_y)$ is infinite. Note that for axisymmetric turbulence, $I = \frac{1}{\pi} \int d^2k\, A(k)/k$, which, as is seen below, may be interpreted as proportional to the correlation scale of the potential function $a(x, y)$.

It is also convenient to define a total correlation scale for the 2D model, λ_{c2}, as the normalized area under the trace of the spectral matrix,

$$\begin{aligned}\lambda_{c2} &= \frac{\int_0^\infty R(x,0)\,dx}{\langle b^2 \rangle} = \pi \frac{S_{xx}^r(k_x = 0) + S_{yy}^r(k_x = 0)}{\langle b^2 \rangle} \\ &= \frac{2\pi \left[\int_0^\infty k^2 A(\boldsymbol{k})\,dk_y\right]_{k_x=0}}{\langle b^2 \rangle},\end{aligned} \tag{9}$$

and if we assume axisymmetry,

$$\lambda_{c2} = \frac{\int [S(k)/k]\,d^2k}{\langle b^2 \rangle}. \tag{10}$$

Each of these expressions follows readily from the basic properties given above.

In the case of one-dimensional (slab) turbulence, the correlation length would not only characterize the length over which the correlation function R is substantial, but would also determine the diffusion coefficient in the field line random walk (Jokipii & Parker 1968):

$$D_{\rm slab} = \frac{\lambda_c}{2} \frac{\langle b^2 \rangle}{B_0^2}, \tag{11}$$

where we define the field line diffusion coefficient in, say, the x-direction, by $D \equiv \langle \Delta x^2 \rangle/(2\Delta z)$.

In contrast, a different length scale, distinct from the correlation scales, enters in applications of 2D turbulence and can be defined for axisymmetric fluctuations as

$$\tilde{\lambda} = \sqrt{\frac{\int d^2k\, S(k) k^{-2}}{\langle b^2 \rangle}} = \sqrt{\frac{\langle a^2 \rangle}{\langle b^2 \rangle}}. \tag{12}$$

This definition differs by a factor $1/\sqrt{2}$ from that in Matthaeus et al. (1995). For most reasonable forms of the spectrum, this length scale is larger than the correlation scale, and for this reason, $\tilde{\lambda}$ has been called the ultrascale. The ultrascale or its analogs appear in theories of self-diffusion in hydrodynamic turbulence (Salu & Montgomery 1977) and diffusion in guiding center plasmas (Taylor & McNamara 1971). The ultrascale also emerges in the description of the 2D contribution to the magnetic field line random walk for the axisymmetric case (Matthaeus et al. 1995),

$$D_{\rm 2D} = \frac{\tilde{\lambda}\sqrt{\langle b^2 \rangle/2}}{B_0}, \tag{13}$$

where it is assumed that the 2D fluctuating field is perpendicular to the mean field. Equation (13) suggests that $\tilde{\lambda}$ can be interpreted as the perpendicular coherence length of the field line random walk (Ruffolo et al. 2004), a typical radius of curvature of the field line trajectory in (x, y), and a typical "island" size of the 2D turbulence (see also Matthaeus et al. 1999). The ultrascale also arises in certain limits of charged particle transport in composite (slab + 2D) turbulence (Matthaeus et al. 2003; Minnie et al. 2007).

For nonaxisymmetric 2D fluctuations, the diffusion coefficients can be different in the x- and y-directions. In analogy to the two correlation scales, let us introduce different ultrascales for different directions, so that equation (13) becomes

$$\begin{aligned}D_x &\equiv \frac{\langle \Delta x^2 \rangle}{2\Delta z} = \frac{\tilde{\lambda}_x \sqrt{\langle b_x^2 \rangle}}{B_0}, \\ D_y &\equiv \frac{\langle \Delta y^2 \rangle}{2\Delta z} = \frac{\tilde{\lambda}_y \sqrt{\langle b_y^2 \rangle}}{B_0},\end{aligned} \tag{14}$$

where

$$\begin{aligned}\tilde{\lambda}_x &\equiv \sqrt{\int \frac{S_{xx}(\boldsymbol{k})}{k_x^2 + (D_y/D_x)k_y^2} d^2k \Big/ \int S_{xx}(\boldsymbol{k})\,d^2k}, \\ \tilde{\lambda}_y &\equiv \sqrt{\int \frac{S_{yy}(\boldsymbol{k})}{(D_x/D_y)k_x^2 + k_y^2} d^2k \Big/ \int S_{yy}(\boldsymbol{k})\,d^2k}.\end{aligned} \tag{15}$$

This reduces to equation (12) for the axisymmetric case. In combination, equations (14) and (15) represent two coupled, implicit equations for D_x and D_y. Ruffolo et al. (2006) have solved these analytically for the case where $A(\boldsymbol{k})$ is constant along ellipses instead of circles, representing axisymmetric turbulence that is stretched in one direction. For a given aspect ratio of the ellipse, ξ, their results imply that

$$\tilde{\lambda}_x = \sqrt{\frac{1+\xi^2}{2}}\tilde{\lambda}_1, \qquad \tilde{\lambda}_y = \sqrt{\frac{1+\xi^2}{2\xi^2}}\tilde{\lambda}_1, \tag{16}$$

where $\tilde{\lambda}_1$ is an ultrascale for the axisymmetric case $\xi = 1$.

It is clear that a number of characteristic length scales can be defined in 2D turbulence (as in 3D). For example, considering axisymmetric 2D magnetic fluctuations, one can define a series of length scales from the moments in k-space of the energy spectrum,

$$\frac{\int d^2k\, S(\boldsymbol{k})k^\beta}{\int d^2k\, S(\boldsymbol{k})}. \tag{17}$$

The $\beta = -2$ moment determines the ultrascale, the $\beta = -1$ moment is proportional to the correlation scale, and $\beta = 2$ is related to the so-called Taylor microscale, which we do not discuss further here (see Batchelor 1970). The sequence of characteristic lengths can be equally well defined in terms of the spectrum $A(\boldsymbol{k})$. One should note that if the correlation function depends parametrically on a single scale, as is the case for a simple exponential, then all the above moments are related to one another in a simple way. But for an arbitrary correlation function, each of the above lengths is completely independent.

4. BEHAVIOR OF THE SPECTRUM AT VERY LARGE SCALES

First let us examine the behavior of $A(\boldsymbol{k})$ as $k \to 0$. We assume axisymmetry and sufficient regularity to permit a Taylor-Maclaurin series at the origin in wavenumber. In order to have finite energy, we demand that $2\pi \int A(k)k^3\, dk < \infty$. Suppose $A(k) \sim k^q$ as $k \to 0$ and therefore that the integrand behaves as $\sim k^{3+q}$ as $k \to 0$. For integrability, we need to have $q > -4$. Developing a power series about $k = 0$ in integer powers of k, we express A as $A(\boldsymbol{k}) \sim a_{-3}/k^3 + a_{-2}/k^2 + a_{-1}/k + a_0 + a_1 k + \ldots$. However, for homogeneous turbulence, $R_{ii}(\boldsymbol{r}) = R_{ii}(-\boldsymbol{r})$ to ensure translational symmetry. By equation (3), S_{ii} is even in $\boldsymbol{k}$, and therefore $A(\boldsymbol{k}) = S_{ii}(\boldsymbol{k})/k^2$ is also an even function of k and its components. Consequently, a_{-3} and a_{-1} must vanish. The first term that can appear is a_{-2}, and if $a_{-2} \neq 0$ then $\langle b^2 \rangle$ is finite, but $\langle a^2 \rangle$ diverges. This is the case of an infinite ultrascale. The requirements imposed so far, finite energy and evenness in $\boldsymbol{k}$, therefore imply that $S(\boldsymbol{k}) = k^2 A(\boldsymbol{k}) = a_{-2} + a_0 k^2 + a_2 k^4 + \ldots$ for small k in axisymmetric 2D turbulence.

5. REQUIREMENTS OF HOMOGENEITY AND SOLENOIDALITY

Following the classical treatment of homogeneous 3D hydrodynamic turbulence, we assume that the second-order moments (spectra and correlation functions) are homogeneous and analytic as $k \to 0$. This is related to, but somewhat stricter than, the expansion in the previous subsection.

The behavior of the spectral tensor near $\boldsymbol{k} = 0$ can then be expressed as

$$S_{ij}(\boldsymbol{k}) = a_{ij} + b_{ijl}k_l + c_{ijlm}k_l k_m + d_{ijlmn}k_l k_m k_n + \ldots. \tag{18}$$

The hydrodynamic property of incompressibility corresponds to the solenoidal property of the primitive field $\boldsymbol{b}$, namely $\nabla \cdot \boldsymbol{b} = 0$. We therefore demand that $k_i S_{ij}(\boldsymbol{k}) = 0 = k_j S_{ij}(\boldsymbol{k})$, for all $\boldsymbol{k}$, which implies that $a_{ij} = 0$. Furthermore, the spectrum is positive definite in the sense that the diagonal elements in principal axis coordinates are energies ≥ 0. Consequently, the spectral tensor is a positive definite quadratic form (by Cramer's theorem) and $\lambda_i^* S_{ij}(\boldsymbol{k})\lambda_j \geq 0$ for an arbitrary complex vector λ_j (Batchelor 1970). If this is to hold at very small k, one requires $\lambda_i^* \lambda_j k_l b_{ijl} \geq 0$. But this must hold for all $\boldsymbol{k}$, and in particular this quantity reverses sign when $\boldsymbol{k} \to -\boldsymbol{k}$. Therefore, $b_{ijl} = 0$. Furthermore, homogeneity implies that $R_{ij}(\boldsymbol{r}) = R_{ji}(-\boldsymbol{r})$, and so the spectrum must satisfy $S_{ij}(\boldsymbol{k}) = S_{ji}(-\boldsymbol{k})$. Thus the diagonal elements must be even functions of the components of $\boldsymbol{k}$. Consequently,

$$\begin{aligned} S_{xx}(\boldsymbol{k}) &= C_{11lm}k_l k_m + O(k^4), \\ S_{yy}(\boldsymbol{k}) &= C_{22lm}k_l k_m + O(k^4). \end{aligned} \tag{19}$$

This implies, from equation (2), that the leading order behavior of the energy spectrum near the origin is given by

$$S(\boldsymbol{k}) = Ck^2 + O(k^4) \text{ as } k \to 0. \tag{20}$$

This corresponds to a vector potential that behaves as $A(\boldsymbol{k}) \sim C + Dk^2 + O(k^4)$ as $k \to 0$.

6. SOME TYPES OF 2D FLUCTUATIONS

6.1. *Reference Case: Typical Slab Case*

A familiar situation is one-dimensional slab turbulence with a fluctuation $\boldsymbol{b}$ perpendicular to a uniform DC mean field B_0 and varying only in that direction. The total magnetic field is $\boldsymbol{B}(z) = (b_x(z), b_y(z), B_0)$. If the two point correlation $R(r) = \langle \boldsymbol{b}(0) \cdot \boldsymbol{b}(r) \rangle$ vanishes asymptotically beyond a certain spatial lag λ_s, then the spectrum (Fourier transform of the two point correlation) for wavenumbers $k < 1/\lambda_s$ must approach a constant, nonzero value. This reasoning leads to canonical forms for the slab power spectrum, such as,

$$\begin{aligned} S(k \leq 1/\lambda_s) &= C\langle b^2 \rangle \lambda_s, \\ S(k > 1/\lambda_s) &= C\langle b^2 \rangle \lambda_s (k\lambda_s)^{-\nu}, \end{aligned} \tag{21}$$

where C is a normalization constant, and λ_s is the slab bendover scale. The power-law index ν at higher wavenumbers is frequently set to 5/3, corresponding to the Kolmogorov spectrum for turbulent fluctuations in the inertial range. The spectrum has the property that $2\int_0^\infty dk\, S(k) = \langle b^2 \rangle$, from which one determines that $C = 1/5$. One can readily show for this spectrum that the correlation scale is

$$\lambda_{cs} = \frac{\int_0^\infty dr\, R(r)}{R(0)} = \frac{\pi}{5}\lambda_s. \tag{22}$$

6.2. *Analogous 2D Spectrum*

The above slab spectrum, with dimensionality equal to the magnetic field squared times the length, has the property that its integral over wavenumber gives the total energy. By analogy, one might adopt a similar form for $\mathcal{E}$, the 2D omnidirectional spectrum, for which $\int \mathcal{E}(k)\, dk = \langle b^2 \rangle$. Therefore, one might consider models in which the 2D omnidirectional spectrum is of the form

$$\mathcal{E}(k) = 2\pi k S(\boldsymbol{k}) = C\langle b^2\rangle\lambda_2 \qquad (\text{for } k \le 1/\lambda_2)$$
$$= C\langle b^2\rangle\lambda_2(k\lambda_2)^{-\nu} \quad (\text{for } k > 1/\lambda_2), \tag{23}$$

where C is a normalization constant, and λ_2, the 2D bendover scale, can be related to the 2D correlation scale. Here, it is understood that $\boldsymbol{k}$ is a 2D wavevector, and the inertial range index is usually $\nu = 5/3$. We require $\nu > 1$ for a finite total fluctuation energy. This spectrum is similar in its asymptotic behavior to what has been used in a number of earlier studies (Qin et al. 2002; Zank et al. 2004; Bieber et al. 1994). [Note that these earlier studies have typically used smooth spectra of the form $S(k) = C'\langle b^2\rangle k^{-1}(1 + k^2\lambda^2)^{-\nu/2}$ for C' a normalizing constant.] By comparison with the results of the last two subsections, or directly by integration and using equations (9) and (10), one can show that this spectrum gives rise to an integrable energy, a divergent correlation scale, and an infinite ultrascale.

6.3. *Other 2D Spectral Forms with One Scale Length*

An axisymmetric turbulent fluctuation (modal) spectrum with an arbitrary power-law behavior near $k = 0$ can be written as

$$S(k) = C\langle b^2\rangle\lambda_2^2(k\lambda_2)^p \qquad (\text{for } k \le 1/\lambda_2)$$
$$= C\langle b^2\rangle\lambda_2^2(k\lambda_2)^{-\nu-1} \quad (\text{for } k > 1/\lambda_2), \tag{24}$$

where C is a dimensionless constant determined by normalization. We examine integral values of p, with reference to the power series discussed in §§ 4 and 5. Note that this spectral form still involves only one scale length, λ_2.

Anticipating some divergences as $k \to 0$, it is convenient to introduce a long-wavelength cutoff at a minimum wavenumber $k_0 = 1/L$. The length L corresponds to a system size, or "box size," at which homogeneity might break down. In the theory of field line random walks, there is an analogous cutoff at small wavenumbers. In that case, L corresponds to the rms fluctuation distance of the random walk over the length of interest, and it appears in a function that tends to zero (only) for $k \lesssim k_0 = 1/L$, which for finite L, eliminates any divergences as $k \to 0$ (Ruffolo et al. 2004). In either case, the limit $L \to \infty$ needs to be examined to approach infinite extent and exact homogeneous symmetry or to understand whether the correlation scale and ultrascale exist according to their usual definitions. Alternatively, one might compute the integrals using a large scale cutoff L and speak of the cutoff dependence of λ_c and $\tilde{\lambda}$.

The case $p = -1$ is considered above, in equation (23), and corresponds to a divergent correlation scale $\lambda_{c2} \sim \lambda_2 \log(L/\lambda_2)$ for a large box of size L. For that case, the ultrascale diverges as $\tilde{\lambda} \sim (\lambda_2 L)^{1/2}$. For $p = 0$, the modal spectrum $S(k)$ is flat at the origin, and $\mathcal{E}(k) \sim k$. The normalization constant C can be determined by direct integration to evaluate $\langle b^2\rangle$, yielding $C = (1/\pi)(\nu - 1)/(\nu + 1)$ for $p = 0$. Now one finds a finite correlation scale $\lambda_{c2} = (2 - 2/\nu)\lambda_2$. However, the ultrascale requires that we examine the integral

$$\tilde{\lambda}^2\langle b^2\rangle = 2\pi\int_{1/L}^{\infty} dk\,\frac{S(k)}{k}$$
$$= 2\pi\int_{1/L}^{1/\lambda_2} dk\,\frac{(C\langle b^2\rangle\lambda_2^2)}{k} + 2\pi\int_{1/\lambda_2}^{\infty} dk\,\frac{S(k)}{k}$$
$$\sim \lambda_2^2 \log(L/\lambda_2) \qquad (\text{for } p = 0), \tag{25}$$

TABLE 1

Correlation Scale and Ultrascale for Axisymmetric 2D Spectra with One Length Scale, λ_2, the 2D Bendover Scale

p	Correlation Scale, λ_{c2}	Ultrascale, $\tilde{\lambda}$
-1	$\sim\lambda_2 \log(L/\lambda_2)$	$\sim\sqrt{\lambda_2 L}$
0	$\sim\lambda_2$	$\sim\lambda_2\sqrt{\log(L/\lambda_2)}$
≥ 1	$\sim\lambda_2$	$\sim\lambda_2$

Note.—$S(k) \sim k^p$ as $k \to 0$.

which diverges as $L \to \infty$. Again, the ultrascale integral is not sensitive to the spectral form in the inertial range.

At the level of $p = 1$, one recovers for the first time a case where both the correlation scale and ultrascale are finite as the box size approaches infinity. However, this case would be ruled out for a strict application of homogeneity, as discussed above. The case $p = 2$ is consistent with asymptotic homogeneity and gives the first result in this sequence that is consistent with those formal requirements. It also gives finite λ_{c2} and finite $\tilde{\lambda}$, as do all higher values of the power-law index p. Given that λ_2 is the only scale length in the parameterized power spectrum (eq. [24]), it is not surprising that when they are finite, both length scales are of order λ_2 (see Table 1).

6.4. *2D Spectral Forms with Two Scale Lengths*

Now let us consider an axisymmetric modal spectrum $S(k)$ that is continuous over three wavenumber ranges defined in terms of two length scales:

$$S(\boldsymbol{k}) = \begin{cases} C\lambda\lambda_2\langle b^2\rangle(k\lambda)^p & (\text{for } k \le 1/\lambda), \\ C\lambda\lambda_2\langle b^2\rangle(k\lambda)^{-1} & (\text{for } 1/\lambda < k \le 1/\lambda_2), \\ C\lambda_2^2\langle b^2\rangle(k\lambda_2)^{-\nu-1} & (\text{for } 1/\lambda_2 < k), \end{cases} \tag{26}$$

as illustrated in Figure 1. Now $S(k)$ is a power law with index p at low wavenumbers up to $1/\lambda$, and there is a régime where the omnidirectional energy spectrum $\mathcal{E}(k) = 2\pi k S(k)$ is constant over $1/\lambda < k \le 1/\lambda_2$, i.e., for scale sizes between the usual 2D bendover scale λ_2 and the larger scale λ. Then the steeper spectrum at high wavenumbers $k > 1/\lambda_2$ may be used to represent the inertial scale of turbulent fluctuations. Note that this spectrum has the same form as equation (24) at both low and high wavenumbers and tends to equation (24) as $\lambda \to \lambda_2$.

There is some physical motivation for the spectral form of equation (26). Observations of magnetic fluctuations in the solar wind indicate an omnidirectional energy spectrum with $\nu \approx 5/3$ in the inertial range (at $k \gtrsim 1/\lambda_2$; Jokipii & Coleman 1968) and approaches a flattening (Hedgecock 1975) or a k^{-1} dependence in the energy-containing range (at $k \lesssim 1/\lambda_2$; Bieber et al. 1993). At the same time, the solenoidal property and homogeneity require $S(k)$ to increase with k at low wavenumbers (see § 5), hence the need for a second break scale λ, which marks the onset of long-wavelength (low-wavenumber) behavior. The omnidirectional energy spectrum $\mathcal{E}(k)$, power spectrum $S(k)$, and reduced spectra are shown in Figure 1, for $p = -1$, 0, 2, and 4, normalized by $C\langle b^2\rangle = 1$ and setting $\nu = -5/3$, $\lambda_2 = 1$, and $\lambda = 100$. Note that $p = -1$ corresponds to no spectral break at $k = 1/\lambda$, $p = 0$ is an intermediate case, and only the higher values $p = 2$ and $p = 4$ are consistent with the solenoidal property and homogeneity.

From the requirement that

$$\langle b^2\rangle = 2\pi\int_0^{\infty} S(k)k\,dk, \tag{27}$$

TABLE 2

Correlation Scale and Ultrascale for Axisymmetric 2D Spectra with Two Separated Length Scales

p	Correlation Scale, λ_{c2}	Ultrascale, $\tilde{\lambda}$
-1	$\sim\lambda_2 \log(L/\lambda_2)$	$\sim\sqrt{\lambda_2 L}$
0......................................	$\sim\lambda_2 \log(\lambda/\lambda_2)$	$\sim\sqrt{\lambda\lambda_2 \log(L/\lambda)}$
≥ 1	$\sim\lambda_2 \log(\lambda/\lambda_2)$	$\sim\sqrt{\lambda\lambda_2}$

Note.—$S(k) \sim k^p$ for $k < 1/\lambda$, where $\lambda \gg \lambda_2$.

we can determine the normalization constant C:

$$C = \left[2\pi\left(1 + \frac{1}{\nu - 1}\right) - 2\pi\frac{\lambda_2}{\lambda}\left(1 - \frac{1}{p+2}\right)\right]^{-1}. \quad (28)$$

Note that if the two length scales are widely separated, $\lambda \gg \lambda_2$, then $2\pi C$ is of order unity. Next we can evaluate the correlation scale for a finite long-wavelength cutoff L as in § 6.3. For $p = -1$, it diverges logarithmically:

$$\lambda_{c2} = 2\pi C\lambda_2 \log(L/\lambda_2) \qquad (\text{for } p = -1); \quad (29)$$

and for $p \geq 0$, we have

$$\lambda_{c2} = 2\pi C\lambda_2\left[\log\left(\frac{\lambda}{\lambda_2}\right) + \frac{1}{p+1} + \frac{1}{\nu}\right] \qquad (\text{for } p \geq 0), \quad (30)$$

which gives $\lambda_{c2} \sim \lambda_2 \log \lambda/\lambda_2$ for the case $\lambda \gg \lambda_2$.

Next let us calculate the ultrascale. For $p = -1$, we obtain $\tilde{\lambda} \sim (\lambda_2 L)^{1/2}$, as in § 6.3. However, for $p = 0$, the form of the divergence is somewhat different, with $\tilde{\lambda} \sim [\lambda\lambda_2 \log(L/\lambda)]^{1/2}$. Finally, with $p \geq 1$, we obtain the nondivergent result

$$\tilde{\lambda}^2 = 2\pi C\left[\lambda\lambda_2\left(1 + \frac{1}{p}\right) - \lambda_2^2\left(1 - \frac{1}{\nu+1}\right)\right] \qquad (\text{for } p \geq 1). \quad (31)$$

For $\lambda \gg \lambda_2$, this tends to $\tilde{\lambda} \sim (\lambda\lambda_2)^{1/2}$. Table 2 summarizes our results for the correlation scale and the ultrascale, which demonstrate that the ultrascale can be distinct from the correlation scale, with a different dependence on the model parameters.

7. METHOD FOR EVALUATING THE ULTRASCALE FROM DATA

The ultrascale $\tilde{\lambda}$ defined in equation (12) is an independent length scale that influences important physics processes, such as the field line random walk and particle diffusion. Nevertheless, unlike the correlation scale, we do not have a very good idea of what values it might take in space and astrophysical plasmas. In contrast to the correlation scale, we are not aware that methods of determination of $\tilde{\lambda}$ from observational data have been investigated. Here we briefly summarize one possible approach and show a numerical test of the scheme.

We assume that the fluctuations we are measuring are purely 2D, superposed on a uniform mean field, as described in § 2. Identifying the mean field direction as $\hat{z}$, and assuming for convenience that measurements of $\boldsymbol{b}(x, y_0)$ are available for many values of x at some arbitrary choice of the y coordinate, y_0, we proceed to examine the flux integral

$$f(\Delta x, x_0) = \int_{x_0}^{x_0+\Delta x} dx'\, b_y(x', y_0) = a(x_0, y_0) - a(x_0 + \Delta x, y_0). \quad (32)$$

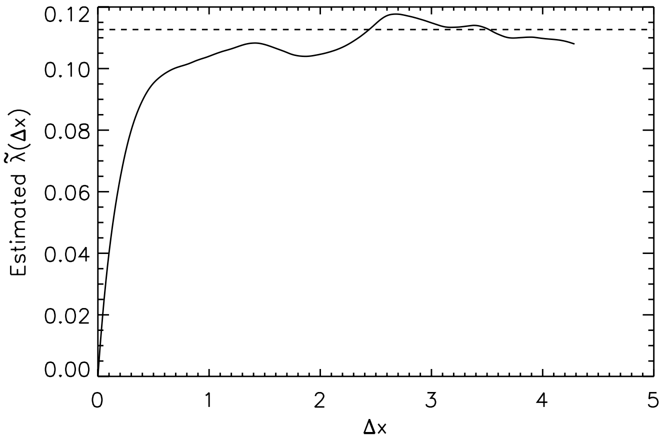

Fig. 2.—Results of evaluation of the ultrascale $\tilde{\lambda}$ from data extracted from a 1024×1024 2D spectral MHD simulation (*solid line*). This confirms that at large Δx, the estimate does tend to the exact value of the ultrascale for the known power spectrum, $\tilde{\lambda} = 0.11267$ (*dashed line*). To calculate the flux integrals, samples of $b_y(x, y_0)$ were taken for 1024 values of x at each of 20 values of y_0. Note that the simulation box size is 2π and the correlation scale is 0.05 for this data set.

We square this value and average over many such intervals of magnetic data, noting that we may freely shift the values of x_0 and y_0, assuming that the turbulence is spatially homogeneous. Therefore, if $\langle\ldots\rangle$ denotes this averaging process, we arrive at a statistical average of the mean square flux integral, which is

$$F(\Delta x) = \left\langle f^2(\Delta x, x_0)\right\rangle = \left\langle (a - a')^2\right\rangle, \quad (33)$$

where $a - a'$ denotes the difference between the values of the potential function at two positions separated by Δx. This is just the second-order structure function of the potential. We now assume that these potential function values become uncorrelated at very large separations, so that

$$\lim_{\Delta x \to \infty} F(\Delta x) = F_\infty = 2\langle a^2\rangle. \quad (34)$$

Under these conditions, we see that

$$\tilde{\lambda} = \sqrt{\frac{F_\infty}{2\langle b^2\rangle}}, \quad (35)$$

which amounts to a simple prescription that might be used to compute values of $\tilde{\lambda}$ from magnetic turbulence data. The main restrictions on this approach appear to be that the sequence of measured mean-square flux integrals must approach a stable limit and that the turbulence is homogeneous and quasi-2D over the requisite scales.

To demonstrate this, we employ synthetic magnetic field data (as in Matthaeus et al. 1995). The data are generated with a controlled spectrum, with $p = 2$, and random phases on an evenly spaced 1024×1024 spatial grid. Furthermore, from direct evaluation of the wavenumber-space definition of $\tilde{\lambda}$ given the known power spectrum, the ultrascale here has the value 0.11267, compared with a correlation scale set to 0.05 (units such that box side is 2π). We then proceed to estimate this value from computation of a series of flux integrals, where we average the values over all available estimates in the selected data sample for each choice of Δx and then plot the results as Δx is varied.

Figure 2 shows the results of this numerical test. Indeed, we see that this method succeeds in this case, tending to within 10%

of the exact value of $\tilde{\lambda}$. We propose that this may form the basis for the estimation of ultrascale values in space and astrophysical contexts, such as the solar wind, where the flux integrals can be directly computed from observations at a single spacecraft, as the plasma passes by, for a range of separations Δx in the plasma frame of reference. It might also be adapted to solar and other astrophysical settings in which spatially resolved vector magnetic field data are available.

8. DISCUSSION AND CONCLUSIONS

We have presented the formal properties of the power spectra and physically important length scales in 2D turbulence models. These can be significant for various applications, since transport coefficients for particle diffusion (Taylor & McNamara 1971), turbulent self-diffusion (Salu & Montgomery 1977), and the field line random walk (Matthaeus et al. 1995) can depend on the above-described low-wavenumber part of the spectrum in certain limits. Furthermore, there is evidence in the heliosphere (Bieber et al. 1994, 1996), Earth's magnetotail (Montgomery 1987; Borovsky et al. 1997), and in theoretical work (Shebalin et al. 1983; Higdon 1984; Goldreich & Sridhar 1995) that 2D turbulence, or turbulence that is nearly 2D, can be dominant in many circumstances, especially when there is a moderate to strong externally supported, or very large scale, mean magnetic field. The present work could be extended to numerically study the anticipated effects on field line and particle transport perpendicular to a mean magnetic field, as low-wavenumber spectral properties are varied.

The results of calculating the correlation length and ultrascale for a varying power-law index p near the origin are summarized in Table 1, for example spectra with a single length scale, and in Table 2, for spectra with two length scales. The computed energy is finite in all cases listed in the tables, but would be infinite for integral values of $p < -1$. For $p = -1$ and a box size L, the ultrascale diverges as $\sqrt{L}$, while the correlation scale diverges weakly as $\log L$. For $p = 0$, the ultrascale is weakly divergent, but the correlation scale is finite and of the order of the bendover scale λ_2. For $p = 1$ or greater, both scales are finite. These three distinct possibilities may all be realizable, since all have finite energy. For the spectra with two length scales, the correlation scale and ultrascale are shown to have distinct dependences on those scales and thus must be considered to be independent of one another.

The present paper hints at several additional issues that will require future study. For example, the formal properties described here can be employed to re-examine standard spectrum models (Bieber et al. 1994; Zank et al. 2004) and possibly to develop and test variations of these. This leads naturally to the desirability of understanding better what observations might have to tell us about the low-wavenumber spectrum and the ultrascale. Extrapolating from the present purely theoretical perspective, one might suppose that two alternatives exist in real astrophysical applications: one would be that the ultrascale exists and is finite, which would place constraints on the low-wavenumber spectra, as we have shown above. The second would be that the spectrum does not admit the behavior needed for a well-defined ultrascale. In such cases it is likely that the macroscopic system size introduces a cutoff that acts in place of the ultrascale with regard to physical issues, such as diffusion. The impact of long-wavelength cutoffs is summarized in Tables 1 and 2. The remaining possibility, that of statistical homogeneity at all scales in an unbounded domain, and with infinite ultrascale, appears to be a mathematical, and not a realistic physical, option.

Furthermore, with regard to connection to real systems, we have presented a technique for determining the ultrascale from observations at a single point as the plasma flows by, demonstrating its accuracy for synthetic data. The implementation of this method to compute the ultrascale from observed magnetic field data would also be valuable. However, we anticipate that the difficulties that usually plague determination of very low wavenumber (or low frequency) spectra will complicate this procedure as well. For example, proper determination of the mean vector magnetic field is anticipated to be an issue, due to very low frequency signals. Furthermore, the methodology will need to determine the presence of additional components beyond the pure 2D ingredient that has been the present emphasis. Finally, since all of the above results are for strictly 2D models, an examination of the generalization of these properties to anisotropic three-dimensional models is suggested.

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